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Yukawa hierarchies from spontaneous breaking of the $SU(3)_L \times SU(3)_R$ flavour symmetry?

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Abstract: The tree level potential for a scalar multiplet of 'Yukawa fields' Y for one type of quarks admits the promising vacuum configuration $\langle Y\rangle \propto \mathrm{diag}(0,0,1)$ that breaks spontaneously $SU(3)_L \times SU(3)_R$ flavour symmetry. We investigate whether the vanishing entries could be lifted to nonvanishing values by slightly perturbing the potential, thus providing a mechanism to generate the Yukawa hierarchies. For theories where at the lowest order the only massless states are Nambu-Goldstone bosons we find, as a general result, that the structure of the tree-level vacuum is perturbatively stable against corrections from scalar loops or higher dimensional operators. We discuss the reasons for this stability, and give an explicit illustration in the case of loop corrections by direct computation of the one-loop effective potential of Yukawa fields. Nevertheless, a hierarchical configuration $\langle Y\rangle \propto \mathrm{diag}(\epsilon', \epsilon, 1)$ (with ϵ', $\epsilon \ll 1$) can be generated by enlarging the scalar Yukawa sector. We present a simple model in which spontaneous breaking of the flavour symmetry can give rise to the fermion mass hierarchies.

Keywords: Beyond Standard Model, Quark Masses and Standard Model Parameters, Spontaneous Symmetry Breaking

Contents

1 Introduction

Fermion family replication represents probably the oldest unexplained puzzle in elementary particle physics, dating back to the discovery of the muon by Anderson and Neddermeyer at Caltech in 1936. With the discovery of all the other second and third generation particles, the puzzle became even more intriguing because fermions with the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ quantum numbers have been found with mass values that span up to five orders of magnitude. Explaining such strongly hierarchical mass patterns requires a more fundamental theory than the Standard Model (SM), and a plethora of attempts in this direction have been tried. In their large majority they basically follow two types of approaches:

(i) The first is to postulate new symmetries under which fermions with the same SM quantum numbers transform differently. The fact that fermion families appear to replicate is then just an illusory feature of the low energy theory, due to our incomplete knowledge of the full set of fundamental quantum numbers. This is, for example, the basic ingredient of the popular Froggatt-Nielsen mechanism [1], in which the hierarchy of the Yukawa couplings follows from a dimensional hierarchy in the corresponding effective Yukawa operators, obtained by assigning to the lighter generations larger values of new Abelian charges.

(ii) A different approach is to assume that the different generations contain exact replica of the same set of states. The gauge invariant kinetic terms of each type of fermions

of the same charge and chirality is then characterized at the fundamental level by a $U(3)$ (flavour) symmetry [2]. This symmetry must be broken: when the breaking is explicit and provided simply by the Yukawa terms we have the SM. However, interesting theoretical attempts have been put forth in which the symmetry is broken spontaneously by vacuum expectation values (vevs) of scalar 'Yukawa' fields, that transform under the various $U(3)$ in such a way that, at the Lagrangian level, the flavour symmetry is exact [3–13].

The first approach basically relies on *ad hoc* assignments of new quantum numbers in order to reproduce qualitatively the observed mass patterns. The other approach, which is the one pursued in this paper, can be considered theoretically more ambitious (as it relies on less *ad hoc* assumptions) although it is by far more challenging than the first one regarding successful model implementations. In order to offer a natural solution to the Yukawa hierarchy, such models should not rely on a hierarchical arrangement of parameters or some tuning between them while, *e.g.*, loop-induced hierarchies would be plausible.

2 Symmetry, invariants, and the tree-level scalar potential

The SM fermions are arranged into triplets of states with the same gauge quantum numbers, and it is then natural to postulate some symmetry group that commutes with the SM gauge group and has three-dimensional representations. The symmetry, however, is not realized in the spectrum, and generally this signals a non invariant ground state yielding spontaneous symmetry breaking (SSB).

The interesting question is which type of SSB, if any, could split the masses of the members of a multiplet and produce the large hierarchies that we observe. A step in this direction was taken in ref. [12] and, in order to introduce the theoretical framework and notation, we will now recall the main results obtained, following ref. [12] in particular.

We do not attempt to build a complete flavour model but rather explore the possibilities of this kind of approaches. We therefore start simply with a pair of flavour triplets of SM fermions with opposite chirality ψ_L^i, ψ_R^j, $(i,\, j = 1, 2, 3)$ and in representations of the gauge group such that their bilinear combination $\bar{\psi}_L^i \psi_R^j$ can be coupled to the Higgs in a gauge invariant way. The largest symmetry of their gauge invariant kinetic term is $U(3)_L \times U(3)_R$ where the first factor acts on the electroweak fermion doublets ψ_L and the second on the weak singlets ψ_R. Here we will concentrate on the semisimple flavour subgroup

$$\mathcal{G}_F = SU(3)_L \times SU(3)_R \tag{2.1}$$

since the fate of the $U(1)$ factors (whether they are broken or contribute to linear combinations of unbroken generators, as e.g. Baryon number) is of no relevance in what follows. We assume that the SM Yukawa term which couples $\psi_{L,R}$ to the Higgs field H originates from a non-renormalizable effective coupling

$$\mathcal{L}_Y = \frac{1}{\Lambda}\, \bar{\psi}_L\, Y\, \psi_R\, H \tag{2.2}$$

that involves a scalar 'Yukawa' field Y (which is, in fact, a matrix in flavour space) and some large scale Λ at which the effective Yukawa operator arises. Invariance of $\mathcal{L}_Y$ under $\mathcal{G}_F$ fixes the following quantum number assignments under $SU(3)_L \times SU(3)_R$:

$$\psi_L \sim (3,1), \qquad \psi_R \sim (1,3), \qquad Y \sim (3,\bar{3})\,. \tag{2.3}$$

If Y acquires a vev, the flavour group $\mathcal{G}_F$ gets spontaneously broken. This of course amounts to interpreting the SM explicit breaking as the result of SSB.

2.1 The $T, A, \mathcal{D}$ invariants

In the following we will denote by Y a generic background field configuration with components of constant value, although sometimes the spacetime dependence will be indicated explicitly, $Y(x)$, to emphasize this is a field (matrix). Configurations that minimize the potential will be instead denoted by $\langle Y \rangle$. To write down the most general renormalizable $\mathcal{G}_F$-invariant potential for Y, and explore its possible ground state configurations and properties, let us consider the characteristic equation for the eigenvalues ξ of the Hermitian matrix $YY^\dagger$

$$\mathcal{P}(\xi) \equiv \det\left(\xi I - YY^\dagger\right) = \xi^3 - T\,\xi^2 + A\,\xi - D^2 = 0\,, \tag{2.4}$$

where $I = I_{3\times 3}$ is the identity matrix in flavour space, and the coefficients are

$$T = \mathrm{Tr}(YY^\dagger) = \sum_i \xi_i\,, \tag{2.5}$$

$$A = \mathrm{Tr}\left[\mathrm{Adj}(YY^\dagger)\right] = \sum_{i>j} \xi_i \xi_j\,, \tag{2.6}$$

$$D^2 = \mathrm{Det}(YY^\dagger) = \prod_i \xi_i\,. \tag{2.7}$$

Being the eigenvalues invariant under group transformations, so are the coefficients of the characteristic equation for $YY^\dagger$, namely its trace T (positive definite and of dimension 2), the trace of its adjugate (or equivalently of the cofactor) matrix A (positive definite and of dimension 4), and its determinant D^2, which is an invariant of dimension 6. However, under special unitary transformations $V_{L,R}$ of $SU(3)_{L,R}$ (with $\mathrm{Det}V_{L,R} = +1$) we have for the determinant of Y: $\mathcal{D} \equiv \mathrm{Det}(Y) \to \mathrm{Det}(V_L Y V_R^\dagger) = \mathcal{D}$, so that $\mathcal{D}$ is also an invariant, but of dimension 3 and thus renormalizable. We conclude that $T(x)$, $A(x)$ and $\mathcal{D}(x)$ are the renormalizable symmetry invariant field combinations from which the scalar potential can be constructed.

In fact, one can show that the most general $SU(3)_L \times SU(3)_R$ invariant potential including nonrenormalizable terms of any dimension, can always be expressed as a function of just the three $T, A, \mathcal{D}$ invariants, that is, it has the form $V(T, A, \mathcal{D}, \mathcal{D}^*) + \mathrm{h.c.}$. This amounts to proving that any invariant term of higher order in Y can be reduced to powers of $T, A, \mathcal{D}$. For determinants of higher powers of Y we have straightforwardly $\mathrm{Det}(Y^m Y^{\dagger n}) = \mathcal{D}^m \mathcal{D}^{*n}$. For trace invariants let us define:

$$T_{2n} = \mathrm{Tr}[(YY^\dagger)^n]\,, \qquad A_{2n} = \mathrm{Tr}[\mathrm{Adj}(YY^\dagger)^n]\,. \tag{2.8}$$

According to this notation, $T = T_2$ and $A = A_2$. It is straightforward to show that

$$T_4 = T_2^2 - 2\,A_2\,, \qquad A_4 = A_2^2 - 2\,T_2\,D^2\,. \tag{2.9}$$

To show that higher order invariants $T_{2n}\,, A_{2n}$ with $n > 2$ can also be written in terms of T_2, A_2, D^2 we can make use of the Cayley-Hamilton theorem, which states that every square matrix of complex numbers satisfies its own characteristic equation. That is, by substituting $\xi \to YY^\dagger$ in $\mathcal{P}(\xi)$ of eq. (2.4), one has the matrix equation:

$$\mathcal{P}(YY^\dagger) = (YY^\dagger)^3 - T_2\,(YY^\dagger)^2 + A_2\,(YY^\dagger) - D^2\,I = 0\,. \tag{2.10}$$

This allows to rewrite $(YY^\dagger)^3$, and thus recursively any other higher power of $YY^\dagger$, in terms of the three fundamental invariants and of $(YY^\dagger)$ and $(YY^\dagger)^2$ which in turn reduce to T_2, A_2, D^2 after taking the trace or after tracing their adjugates and using eq. (2.9).

2.2 Scalar potential and tree-level vacua

A necessary condition to ensure that the observed hierarchy of the SM Yukawa couplings is reproduced, is that, at the SSB minimum:

$$\langle D\rangle^{1/3} \ll \langle A\rangle^{1/4} \ll \langle T\rangle^{1/2}\,, \tag{2.11}$$

where $D = |\mathcal{D}|$. The first goal is then to construct a scalar potential which naturally has such minimum. In terms of the $T, A, \mathcal{D}$ invariants the most general renormalizable potential for Y can be written as [12][1]

$$V_0 = V_T + V_A + V_{\mathcal{D}}\,, \tag{2.12}$$

with

$$V_T = \lambda\left[T - \frac{m^2}{2\lambda}\right]^2, \tag{2.13}$$

$$V_A = \lambda_A A\,, \tag{2.14}$$

$$V_{\mathcal{D}} = \tilde{\mu}\,\mathcal{D} + \tilde{\mu}^*\,\mathcal{D}^* = 2\,\mu\,D\,\cos\phi_{\mathcal{D}}\,. \tag{2.15}$$

We assume that all the Lagrangian parameters are evaluated at the scale Λ, which can be identified with that in eq. (2.2). V_T in eq. (2.13) contains the two renormalizable invariants constructed from the trace, $V_T = \lambda T^2 - m^2 T$, plus an irrelevant constant. We require $\lambda > 0$ and $m^2 > 0$ in order to have a potential bounded from below and to trigger SSB. The parameter λ_A which multiplies A can be either positive or negative, and we need to consider both possibilities. The last equality in eq. (2.15) is obtained after defining $\tilde{\mu} = \mu e^{i\delta}$ with $\mu \equiv |\tilde{\mu}|$, $\mathcal{D} = e^{i\varphi(x)}D$, and $\phi_{\mathcal{D}}(x) = \varphi(x) + \delta$.

Let us now seek the most general form for the vev of the scalar field. A generic 3×3 matrix of (complex) constant background fields Y has 9 moduli and 9 phases, and by means of an $SU(3)_L \times SU(3)_R$ rotation (corresponding to 3+3 moduli and 5+5 phases) can always

[1] As long as $\langle H^\dagger H\rangle/\Lambda^2 \ll 1$ the coupling with the Higgs, $H^\dagger H T$, can be omitted from eq. (2.12). Regarding the effects of such coupling on the Higgs potential, electroweak symmetry breaking at the correct scale would require a certain degree of fine-tuning in the term $H^\dagger H\left(\langle T\rangle - \mu_H^2\right)$.

be brought into diagonal form $Y^{(d)} = \mathrm{diag}(Y_{11}, Y_{22}, Y_{33})$. Since the $SU(3)_{L,R}$ diagonal generators λ_3 and λ_8 commute with $Y^{(d)}$, this matrix is invariant under the subgroup $U(1)_{(\lambda_3)_{L+R}} \times U(1)_{(\lambda_8)_{L+R}}$ and therefore, out of the initial 9 phases, only $10 - 2 = 8$ can be removed by flavour rotations. Without loss of generality we can then choose a basis in which the background classical field has the form

$$Y = \frac{1}{\sqrt{2}} \mathrm{diag}(R_{11}, R_{22}, R_{33} + iJ_{33}) \,, \tag{2.16}$$

where R_{ii} and J_{ii} are real scalar fields.

Regarding the value of Y that minimizes the potential in eq. (2.12), that is, the tree-level vev of Y, from eq. (2.15) we immediately see that if $\langle D \rangle \neq 0$ then $V_{\mathcal{D}}$ is minimized when $\cos\phi_{\mathcal{D}} = -1$ (by $\langle \varphi \rangle = \pi - \delta$), so that we can restrict our analysis to $D \geq 0$ in what follows. If instead $\langle D \rangle = 0$ at the minimum of the potential, the phase of $\langle \mathcal{D} \rangle$ is undetermined (and φ has a flat potential). This allows us to set $\langle \varphi \rangle = \pi - \delta$ and search for the minimum around the configuration

$$V_{\mathcal{D}}^{\min} \equiv V_D = -2\mu D, \tag{2.17}$$

$$\langle Y \rangle = \frac{1}{\sqrt{2}} \mathrm{diag}(R_{11}, R_{22}, R_{33}) \,, \tag{2.18}$$

where, with a slight abuse of notation, we have denoted with R_{33} the modulus $\sqrt{2}|Y_{33}|$. From eq. (2.13) we immediately see that V_T is minimized on the surface $\langle T \rangle = m^2/(2\lambda)$ of the sphere in the eighteen dimensional parameter space. Note that since we must require $m^2/(2\lambda) \lesssim \Lambda^2$ in order to explain e.g. the value of the top-quark Yukawa coupling[2], then a perturbative $\lambda < 1$ implies $m^2 < \Lambda^2$, consistently with the effective theory treatment. Concerning A and D, they are both maximized for symmetric vacua $\langle Y \rangle \propto \mathrm{diag}(1, 1, 1)$ and their minimum value is zero. To ensure $\langle D \rangle = 0$, at least one entry in $\langle Y \rangle$ must vanish, while for $\langle A \rangle = 0$ two entries must vanish, e.g. $\langle Y \rangle \propto \mathrm{diag}(0, 0, 1)$. Which particular minimum on the surface of constant $\langle T \rangle$ is selected depends on the sign and value of λ_A and on the value of μ. Following [12] we recall below which types of SSB minima can occur and under which conditions.

(i) When $\lambda_A < 0$ we have to require $|\lambda_A| < 3\lambda$ in order that the potential remains bounded from below. A is maximized for symmetric vacua (eq. (2.19) below) and since $V_A = \lambda_A A$ is negative this is the favoured configuration. D is also maximized for symmetric vacua so that the negative value of $V_{\mathcal{D}}$, eq. (2.17), further lowers the minimum. The symmetric vev

$$\langle Y \rangle^s = v_s \, \mathrm{diag}(1, 1, 1), \tag{2.19}$$

(where v_s is given in eq. (18) of [12] with $\lambda' = -\lambda_A$) corresponds, however, to non-hierarchical Yukawa couplings yielding $\langle T \rangle \approx \langle A \rangle^{1/2} \approx \langle D \rangle^{2/3} = v_s^2$.

[2]For example, at a cutoff scale $\Lambda \sim (10^9 - 10^{12})$ GeV we have $y_t = \langle Y_{33} \rangle / \Lambda \sim 0.6 - 0.5$ [14]. In particular, this justifies neglecting contributions to the Yukawa terms eq. (2.2) of dimension higher than five.

(ii) When $\lambda_A > 0$, $V_A = \lambda_A A$ is always positive and minimized for $\langle A \rangle = 0$, which favours (hierarchical) vacua with two vanishing entries

$$\langle Y \rangle^h = v_h \, \mathrm{diag}(0,\, 0,\, 1)\,. \tag{2.20}$$

with $v_h = m/\sqrt{2\lambda}$. Given that $V\left(\langle Y \rangle^h\right) = 0$, this configuration is selected as long as the potential in the symmetric direction has a positive definite value $V\left(\langle Y \rangle^s\right) > 0$ in spite of a possible negative contribution from V_D. This occurs as long as (see [12] for details)

$$\frac{\mu^2}{m^2} < 2\lambda \left[\left(4 + \frac{\lambda_A}{\lambda}\right)^{3/2} - \left(8 + 3\frac{\lambda_A}{\lambda}\right) \right]. \tag{2.21}$$

In this case $\langle T \rangle = v_h^2$ while $\langle D \rangle = \langle A \rangle = 0$, which represents a promising first approximation to the realistic hierarchy, as in eq. (2.11).

Thus, the tree level analysis indicates that the most general renormalizable $\mathcal{G}_F = SU(3)_L \times SU(3)_R$ invariant potential admits two types of SSB vevs, that lead to the two symmetry breaking patterns $\mathcal{G}_F \to H_s$ and $\mathcal{G}_F \to H_h$ respectively with little groups:

$$H_s = SU(3)_{L+R}\,, \tag{2.22}$$

$$H_h = SU(2)_L \times SU(2)_R \times U(1)_{(\lambda_8)_{L+R}}\,. \tag{2.23}$$

Before concluding this section let us recall some jargon specific to SSB problems, as well as some general results. The largest subgroup $H \subset \mathcal{G}_F$ that leaves invariant some background field configuration $\langle Y \rangle$ is called the *little group* of $\langle Y \rangle$. In particular, the little groups $H_{s,h}$ in eqs. (2.22)-(2.23) are *maximal little groups* in the sense that none is contained in the other or in another little group of $\mathcal{G}_F$. Acting with group elements in $\mathcal{G}_F/H$ on $\langle Y \rangle$ while keeping its 'length' ($\langle T \rangle$ in our case) fixed, one obtains the *orbit* of $\langle Y \rangle$. If in the neighbourhood of $\langle Y \rangle$ all other background configurations have the same little group H, the collection of their orbits is called an *open stratum* (or *dense stratum*). An example of a configuration with orbit belonging to an open stratum is the general form $\langle Y \rangle \sim \mathrm{diag}(a, b, c)$ corresponding to eq. (2.18), with little group[3]

$$H_{abc} = U(1)_{(\lambda_3)_{L+R}} \times U(1)_{(\lambda_8)_{L+R}}\,. \tag{2.24}$$

The boundaries of an open stratum are the closed strata, which contain orbits in the neighborhood of which there are other configurations with different little groups. If $H \subset \mathcal{G}_F$ is a maximal little group, then the corresponding $\langle Y \rangle$ is in a closed stratum. If the closed stratum has only one orbit, this orbit is a stationary point of any smooth real invariant functions of $\langle Y \rangle$ (Michel-Radicati theorem [15]).

In our case, the boundaries of the open stratum of the general background field configuration in eq. (2.18) are $\langle Y \rangle^{s,h}$ in eqs. (2.19)-(2.20) and, in agreement with Michel's theorem [17], their little groups $H_{s,h}$ [eqs. (2.22)-(2.23)] are the *maximal stability groups* of the most general fourth order function of the invariants (the tree-level potential).

[3] Considering the full initial group of invariance $U(3)^2 = U(1)_B \times U(1)_{L-R} \times SU(3)^2$ one readily recognizes that the full little group of continuous transformations for the Yukawa configuration eq. (2.18) is $= U(1)_B \times U(1)_{(\lambda_3)_{L+R}} \times U(1)_{(\lambda_8)_{L+R}}$ which are linear combinations of the three baryon flavour symmetries $U(1)_{B_i}$ (eventually broken to $U(1)_B$ by inter-generational quark mixing).

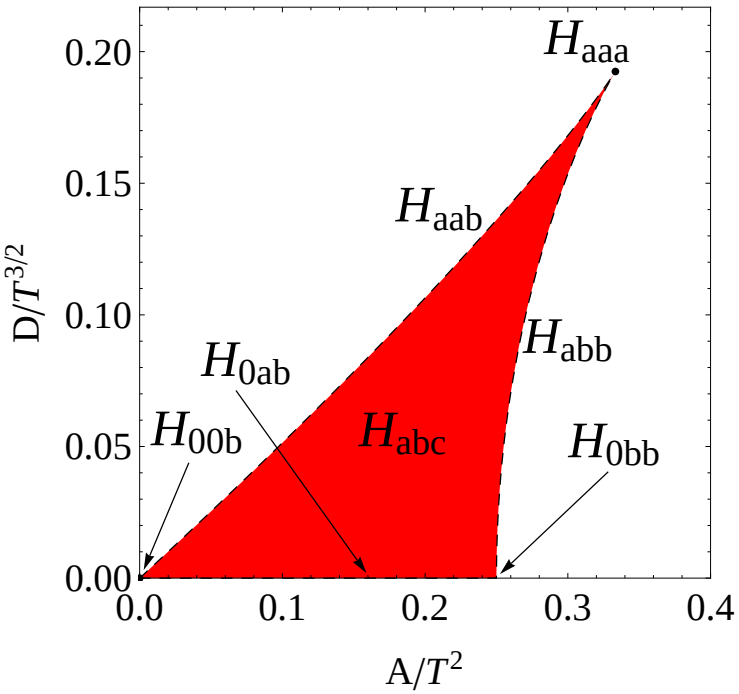

Figure 1. The orbit space for the group $SU(3)_L \times SU(3)_R$ broken by a field Y in the bi-fundamental representation, plotted in the plane $(A/T^2, D/T^{3/2})$ with an arbitrary nonzero T. The H's labeling the points in the plot denote the little groups of the corresponding field configurations $\langle Y \rangle$. $H_{aaa} = H_s$ and $H_{00b} = H_h$ are the maximal little groups of $\langle Y \rangle \sim \mathrm{diag}(a, a, a)$ and $\langle Y \rangle \sim \mathrm{diag}(0, 0, b)$.

As we have seen, although the potential is a function of many scalar fields (Y has 18 degrees of freedom) the fact that only the 3 invariants T, A and D enter, simplifies greatly the minimization problem. A further simplification follows from the observation [16] that the orbit space can be described in a compact way by T (which is nonzero in any symmetry breaking vacuum) and by the two dimensionless ratios of invariants $r_A \equiv A/T^2$ and $r_D \equiv D/T^{3/2}$, which are respectively bounded within the intervals $0 \leq r_A \leq 1/3$ and $0 \leq r_D \leq 1/(3\sqrt{3})$. For given fixed values of r_A and r_D, the potential is a function of T only, with some minima $\langle T \rangle (r_A, r_D)$. The global minimum is then the deepest of such minima in the whole range (r_A, r_D). Such an orbit space does not depend on the details of the potential, but only on the group structure and on the representation of the scalar fields. For our case the orbit space is plotted in figure 1 in which, for instance, H_{aaa} refers to the symmetric little group $H_s = SU(3)_{L+R}$ and labels the field configuration $\langle Y \rangle \sim \mathrm{diag}(a, a, a)$, while a generic point H_{abc} labels the field configuration $\langle Y \rangle \sim \mathrm{diag}(a, b, c)$. For our labels we keep the ordering $a < b < c$, so that the plot gives also information on the relative size of the $\langle Y \rangle$ entries. Note that the point labeled H_{0bb} belongs to the same stratum as H_{abb}, while the point H_{0ab} belongs to the same stratum as H_{abc}. Regarding H_{aab} and H_{abb}, although they label disconnected regions in orbit space, they correspond to the same little group, that is $SU(2)_{L+R} \times U(1)_{(\lambda_8)_{L+R}}$, and so they belong to the same

stratum. Following the method proposed in ref. [16], the little groups of the global minima of V_0 can also be determined through an analysis of equipotential lines in the orbit space. The result is that for $\lambda_A < 0$ the minimum of V_0 always is at $H_{aaa} = \big(1/3, 1/(3\sqrt{3})\big)$, while for $\lambda_A > 0$ the minimum can be at $H_{00b} = (0,0)$ or at H_{aaa}, depending on the values of the parameters (i.e. if eq. (2.21) is satisfied or not). This of course coincides with our previous findings.

The question now is if, as suggested by Michel's conjecture [17], the one-loop effective potential can only have minima with the same maximal little groups $H_{s,h}$ as the tree-level potential, or if phenomenologically more appealing minima with the generic little group H_{abc} are possible. On one hand, if a phenomenologically acceptable minimum had corresponded to a maximal little group, it would have been easy to model a potential leading to it, as this is facilitated by the group structure which makes such symmetry breaking natural. On the other hand, seeking for a symmetry breaking pattern with non-maximal little group is harder but might offer some deeper insight into the dynamics producing such breaking. Be that as it may, as we will discuss in the next two sections, although counter-examples to Michel's conjecture are known to exist [18], in our case $H_{s,h}$ remain stable with respect to perturbative effects from loop corrections or from operators of higher dimension.

3 Sequential breaking

The symmetric solution $\langle Y\rangle^s = v_s\,\mathrm{diag}(1,1,1)$ in eq. (2.19), which yields non-hierarchical Yukawa couplings, is phenomenologically uninteresting and we do not consider it (although it might be of interest in the neutrino sector). The solution $\langle Y\rangle^h = v_h\,\mathrm{diag}(0,0,1)$ in eq. (2.20) appears instead as a promising first approximation to the observed Yukawa hierarchies, but is tenable only if the two vanishing entries can be lifted to small nonzero values by some effect. In this section we want to consider the possibility of obtaining a vacuum $\langle Y\rangle = v_Y\,\mathrm{diag}(\epsilon',\epsilon,1)$ with the non-maximal little group $H_{abc} \subset H_h$, from a small perturbation of the tree-level vacuum $\langle Y\rangle^h$ in eq. (2.20). Such perturbation could be provided, for instance, by one-loop corrections to the effective potential V_{eff} (as hypothesized in ref. [12]) or by higher dimensional operators. We will show that in general these possibilities cannot be realized.

Let us focus for the moment on the symmetry breaking pattern $\mathcal{G}_F \to H_h$, *i.e.* the minimum is $\langle Y\rangle^h = v_h\,\mathrm{diag}(0,0,1)$ with $v_h = m/\sqrt{2\lambda}$. There are $n_G = 8+8$ generators in $\mathcal{G}_F$, and $n_H = 3+3+1 = 7$ unbroken generators in the little group H_h. Nine generators are thus broken and accordingly we find in the spectrum $n_b = 9$ massless Nambu-Goldstone Bosons (NGB). The remaining nine states are arranged in multiplets of H_h with masses:

$$m_1^2 = 4\lambda v_h^2\,, \qquad \text{1 state}, \tag{3.1}$$

$$m_\pm^2 = \lambda_A v_h^2 \pm \mu v_h\,, \qquad 4+4 \text{ states}. \tag{3.2}$$

Assuming that upon minimization of the one-loop effective potential a ground state with little group $H' \subset H_h$ with $n_{H'} < n_{H_h}$ unbroken and $n_{b'} > n_b$ broken generators is obtained, would then imply that out of the 9 massive states, $n_{H_h} - n_{H'}$ have to become the

new massless NGB. The required cancellation between the tree-level and loop mass contributions do not conform to the perturbative approach to this problem. In other words, the zeroth-order mass relations determined by the unbroken subgroup of the symmetry group should persist also at higher orders [19]. More precisely, a theorem proved long ago by Georgi and Pais [20] states that a reduction of the tree-level vacuum symmetry via radiative corrections can only occur if there are additional massless bosons in the tree approximation.[4] This can be seen in the following way: the Goldstone theorem states that for each generator $\mathcal{T}^a$ of a continuous symmetry acting nontrivially on the vacuum there is a massless scalar, that is:

$$\mathcal{T}^a \cdot \langle Y \rangle \neq 0 \implies M^2 \cdot \mathcal{T}^a \cdot \langle Y \rangle = 0\,, \tag{3.3}$$

where M^2 is the second derivative of V with respect to the fields evaluated at $Y = \langle Y \rangle$. Following ref. [20], let us now write $M^2 = M_0^2 + \delta M^2$ and $\langle Y \rangle = \langle Y \rangle_0 + \delta\langle Y \rangle$ where M_0^2 and $\langle Y \rangle_0$ are obtained from minimization of V_0, while δM^2 and $\delta\langle Y \rangle$ are the perturbations induced by higher order corrections to the potential. Stepwise breaking implies that there must be some generator $\mathcal{T}^a$ in the little group H of the tree-level vacuum that is not in H', that is:

$$\mathcal{T}^a \cdot \langle Y \rangle_0 = 0, \qquad \text{and} \qquad \mathcal{T}^a \cdot \langle Y \rangle \neq 0\,. \tag{3.4}$$

At first order in the corrections, eq. (3.3) becomes

$$M_0^2 \cdot \mathcal{T}^a \cdot \langle Y \rangle_0 + \delta M^2 \cdot \mathcal{T}^a \cdot \langle Y \rangle_0 + M_0^2 \cdot \mathcal{T}^a \cdot \delta\langle Y \rangle = 0\,. \tag{3.5}$$

The first two terms in this equation vanish because of the first equation in (3.4), and the vanishing of the last term then implies additional massless scalars, that at the tree level are not NGB. In this case, a further breaking of the symmetry by higher order effects can simply transform some of these additional massless states into NGB. A well known example is the Coleman-Weinberg (CW) potential [21] in which the scale invariance at the tree level implies (non-Goldstone) massless states in the lowest order approximation. The breaking of the symmetry at one loop then transforms some massless states into NGB and gives mass to the remaining ones.

A crucial point in our particular case is that the extra massless scalars in the tree approximation should appear *naturally*, *e.g.* due to some extended accidental symmetry, because of some larger symmetry of the tree-level vacuum [22] or due to renormalization group evolution of parameters (so that these massless scalars appear at some particular renormalization scale). For example, it is technically possible to tune the mass m_-^2 in eq. (3.2) to zero by setting $\lambda_A m = \mu\sqrt{2\lambda}$, and this would result in four additional massless states at the tree level. To verify if this condition allows for a further breaking of the symmetry by loop corrections, we have carried out a numerical minimization of the $SU(3)_L \times SU(3)_R$ one-loop effective potential given in the Appendix. In the general case with no additional massless states, the parameter space of the effective potential remains divided into two regions corresponding to the two vacuum structures $\langle Y \rangle^h = v_h\,\mathrm{diag}(0,0,1)$

[4] Perhaps a bit surprisingly, this applies also to radiative breaking of discrete symmetries like CP [20].

and $\langle Y\rangle^s = v_s\,\mathrm{diag}(1,1,1)$ with a boundary that is still given with a good approximation by condition eq. (2.21). By setting $\lambda_A m = \mu\sqrt{2\lambda}$ one obtains that this condition is never satisfied, the vacuum remains in the symmetric configuration $\langle Y\rangle^s$ with little group H_s and no further breaking occurs.

Although there is no natural way to forbid the two terms V_A and $V_{\mathcal{D}}$ in the potential, additional massless states would be obtained by setting by hand $\mu \to 0$ and $\lambda_A \to 0$ in the tree-level potential eq. (2.12). This is because the surviving part V_T, eq. (2.13), has an accidental symmetry $SO(18)$ that is much larger than $SU(3)_L \times SU(3)_R$. The minimum condition $T = \sum_i (YY^\dagger)_{ii} = m^2/(2\lambda)$ fixes the radius of the eighteen-dimensional hypersphere, leaving the vacuum symmetry $SO(17)$ of the hypersurface which is broken by a choice of the vacuum direction. We thus obtain seventeen massless bosons: the nine NGB of the broken $SU(3)_L \times SU(3)_R$ plus $4+4$ additional massless states corresponding to the eigenvalues in eq. (3.2). Clearly, in this case since the symmetry of the full theory is just $SU(3)_L \times SU(3)_R$ of the Yukawa operator eq. (2.2), it is conceivable that corrections due to other interactions (as the ones that give rise to the effective Yukawa operator, or $SU(3)_L \times SU(3)_R$ gauge interactions if the flavour symmetry is gauged) could eventually yield a stepwise breaking. We will discuss further this possibility in the next section.

In conclusion, lifting the tree-level vev $\langle Y\rangle^h = v_h\,\mathrm{diag}(0,\,0,\,1)$ to a vev with hierarchical components $\langle Y\rangle = v_Y\,\mathrm{diag}(\epsilon',\,\epsilon,\,1)$ would require at least the symmetry reduction $SU(2)_L \times SU(2)_R \times U(1)_{(\lambda_8)_{L+R}} \to U(1)_{(\lambda_3)_{L+R}} \times U(1)_{(\lambda_8)_{L+R}}$. If, as in the case we are considering, there are no additional tree-level massless states, the $n_{H_h} - n_{H_{abc}} = 5$ new NGB cannot appear at the loop level, and thus no stepwise reduction of the symmetry can occur. We will mention in Section 4 possible loopholes to this conclusion.

Concerning the possibility that corrections from higher dimensional operators [8] could provide the sequential symmetry breaking we are looking for, we can again rely on the Goldstone theorem to formulate the necessary conditions for this to happen. Including higher dimensional terms, we can write the scalar potential as

$$V(p_i, \bar{\Lambda}) = V_0(p_i) + \sum_k \frac{1}{\bar{\Lambda}^k}\,\mathcal{O}^{4+k}\,, \tag{3.6}$$

where $V_0(p_i)$ is the tree-level potential, eq. (2.12), which depends on the parameters $\{p_i\} = \{m,\,\mu,\,\lambda,\,\lambda_A\}$ and $\mathcal{O}^{4+k}$ represents the set of operators with dimension $4+k$ ($k \geq 1$) that are perturbatively suppressed by powers of the high energy scale $\bar{\Lambda}$. Eq. (3.5) translates straightforwardly into

$$M_0^2(p_i) \cdot \mathcal{T}^a \cdot \delta\langle Y\rangle = 0\,. \tag{3.7}$$

where $\delta\langle Y\rangle$ depends on $\bar{\Lambda}$. Stepwise breaking implies that $\mathcal{T}^a \cdot \delta\langle Y\rangle$ is non vanishing, which in turn requires additional non NGB massless states in the tree level mass matrix $M_0^2(p_i)$. In the absence of these states, we can then conclude that also operators of higher dimension are unable to induce perturbatively sequential symmetry breaking.

4 Generalization

We have seen that the maximal little groups $H_{s,h}$ of the stationary points of the tree-level potential, eq. (2.12), are stable against corrections to the effective potential induced by loops or higher dimensional operators. Thus, hierarchical minima with the observed structure $\langle Y\rangle = v_Y\,\mathrm{diag}(\epsilon',\,\epsilon,\,1)$ cannot simply occur as a perturbation from $\langle Y\rangle^h = v_h\,\mathrm{diag}(0,\,0,\,1)$. However, ref. [12] found that, if the one-loop potential were to have terms like $A\log A$ and $D\log D$, they could produce global minima different from $\langle Y\rangle^h$ or $\langle Y\rangle^s$ in eqs. (2.19)-(2.20). Also, a potential that indeed can have a hierarchical minimum $\langle Y\rangle = v_Y\,\mathrm{diag}(\epsilon',\,\epsilon,\,1)$ is given in appendix B.2 of ref. [11]. Such examples show that there is no group theory obstruction to finding potentials with minima that have little groups different from the maximal ones $H_{s,h}$.

We will now show that under some general conditions the smallest little group preserved by the minima of smooth functions of the $SU(3)_L \times SU(3)_R$ invariants is $SU(2)_{L+R} \times U(1)_{(\lambda_8)_{L+R}}$. This corresponds, in the diagonal basis, to the structure $\langle Y\rangle \sim \mathrm{diag}(a, a, b)$, which can only allow for a partial hierarchy. In contrast with the results presented in Section 3, the present argument does not rely on perturbative expansions, and can be applied to more general classes of effects.

Let us consider from now on a generic smooth potential function $V(T, A, D)$, where we take $\mathcal{D}$ real from the start and, without loss of generality, the invariants are written in the real diagonal basis for Y, as in eq. (2.18). The minimization equations are ($i = \{1, 2, 3\}$)

$$\frac{\partial V}{\partial T}\frac{\partial T}{\partial \xi_i} + \frac{\partial V}{\partial A}\frac{\partial A}{\partial \xi_i} + \frac{\partial V}{\partial D}\frac{\partial D}{\partial \xi_i} = 0\,, \tag{4.1}$$

where, as above, the ξ_i are the eigenvalues of $\langle YY^\dagger\rangle$ (i.e. $\langle YY^\dagger\rangle = \mathrm{diag}(\xi_1, \xi_2, \xi_3)$ in our specific basis). We have

$$\frac{\partial T}{\partial \xi_i} = 1\,, \qquad \frac{\partial A}{\partial \xi_i} = T - \xi_i\,, \qquad \frac{\partial D}{\partial \xi_i} = \frac{1}{2}\,\frac{D}{\xi_i}\,, \tag{4.2}$$

so that (4.1) leads to the quadratic equation in ξ_i:

$$\frac{\partial V}{\partial A}\,\xi_i^2 - \left[\frac{\partial V}{\partial T} + \frac{\partial V}{\partial A}\,T\right]\xi_i - \frac{D}{2}\,\frac{\partial V}{\partial D} = 0\,. \tag{4.3}$$

Thus, at the extremal point, the three ξ_i eigenvalues should be solutions of the same equation above which, being quadratic, can only have at most two different roots. It is in this sense that we expect generically to have a little group at least as large as $SU(2)_{L+R} \times U(1)_{(\lambda_8)_{L+R}}$. From this, it follows that regardless of the type of potential function V, the only way to obtain three distinct values ξ_1, ξ_2, ξ_3 is to make zero the three coefficients in eq. (4.3). In this particular case the minimization equations reduce to:

$$\frac{\partial V}{\partial T} = 0\,, \quad \frac{\partial V}{\partial A} = 0\,, \quad \frac{\partial V}{\partial D} = 0\,. \tag{4.4}$$

We could have anticipated this: the secular equation (2.4) gives a univocal correspondence (modulo trivial permutations) between its coefficients and its solutions (that are physically

acceptable in the real domain). Then, when all three ξ_i's are different, we can use T, A, D instead of the ξ_i to describe field space, and as we are interested in a fully hierarchical pattern of Yukawas, we are precisely in that situation.[5] In all other cases, when one of the three equations in (4.4) is not satisfied, the vacuum will have at most the partial hierarchy $\langle Y\rangle \sim \mathrm{diag}(a, a, b)$.

We can now understand from another point of view the negative result found in the previous section in our search for a hierarchical minimum by perturbing over the tree-level minimum $\langle Y\rangle^h = v_h\ \mathrm{diag}(0, 0, 1)$. Writing the potential as $V = V_0 + \Delta V$, with the tree-level potential V_0 as in (2.12) and ΔV representing some small correction, either from loop effects or higher order operators, the minimization equations (4.4) become

$$\begin{aligned} 2\lambda T - m^2 + \frac{\partial \Delta V}{\partial T} &= 0\ , \\ \lambda_A + \frac{\partial \Delta V}{\partial A} &= 0\ , \\ -2\mu + \frac{\partial \Delta V}{\partial D} &= 0\ . \end{aligned} \tag{4.5}$$

Let us write $\langle T\rangle = \langle T\rangle_0 + \delta\langle T\rangle$, where $\langle T\rangle_0$ is the value of T at the tree level minimum and $\delta\langle T\rangle$ a shift in this value due to ΔV, with analogous notations for A and D. The tree level minimum corresponding to $\langle Y\rangle^h$ has $\langle T\rangle_0 \neq 0$ and $\langle A\rangle_0 = \langle D\rangle_0 = 0$. Now if one tries to analyze by means of equations (4.5) how this minimum can be perturbed by the correction ΔV, the first key observation is that the tree-level minimum does not satisfy the tree-level form (i.e. with ΔV removed) of all these equations, but only the first one, with $\langle T\rangle_0 = m^2/(2\lambda)$. The shift $\delta\langle T\rangle$ would then be obtained as

$$2\lambda\delta\langle T\rangle = -\left.\frac{\partial \Delta V}{\partial T}\right|_0\ , \tag{4.6}$$

with the subscript 0 indicating that the derivative is evaluated at the tree-level minimum. However, the two other minimization equations are not consistent unless μ and λ_A are suppressed to at least the order of the perturbation ΔV, perhaps being zero. As already noticed in section 3, if μ and λ_A are in fact zero, there are 17 massless states at tree-level, and this clears up the difficulties with Georgi-Pais' theorem [20]. In the diagonal basis, the minimum condition reads $T = \xi_1 + \xi_2 + \xi_3 = m^2/(2\lambda)$ which is satisfied by points on a hyperspherical surface of equivalent $\{\xi_i\}$ vevs, including hierarchical ones. Which vev is eventually selected then would depend on radiative corrections (or on other higher order effects). Nevertheless, the one-loop potential does not resolve this degeneracy: when $\mu = \lambda_A = 0$, ΔV is a function of the T invariant only, *i.e.* no potential for A and D is generated at one-loop. This can be checked explicitly using the results presented in the Appendix, which calculates the contributions to the potential from scalar self-interactions. In principle, interactions of the Yukawa fields with other sectors of the theory could give

[5] In ref. [12], vev structures $\langle Y\rangle = v_Y\ \mathrm{diag}(\epsilon', \epsilon, 1)$ were obtained by minimizing separately two functions respectively of D and A, that included their logs, and assuming that the minimum of the two separate functions gave the global minimum. That procedure is in fact incorrect. By minimizing numerically the full logarithmic potential we have verified that global minima with $\epsilon' = \epsilon$ are obtained.

additional contributions that could change this picture. This, however, would require introducing a significant model-dependence.

Let us examine next the structure of a potential tailored to give any desired pattern of R_{ii}. Constructing such generic potential is in fact straightforward:[6]

$$V \equiv \lambda \left(T - \langle T\rangle\right)^2 + \frac{1}{\Lambda^4}\left(A - \langle A\rangle\right)^2 + \frac{1}{\Lambda^2}\left(D - \langle D\rangle\right)^2 \ , \tag{4.7}$$

where the values of $\langle T\rangle, \langle A\rangle, \langle D\rangle$ correspond to the chosen values of ξ_i according to eqs. (2.5)-(2.7). Omitting an unimportant constant, we can split $V = V_0 + \Delta V$ in the usual renormalizable part

$$V_0 = \lambda T^2 - m^2 T + \lambda_A A - 2\mu D \ , \tag{4.8}$$

plus the correction ΔV, which contains the following two $d = 8$ and $d = 6$ terms:

$$\Delta V = \frac{1}{\Lambda^4} A^2 + \frac{1}{\Lambda^2} D^2 \, . \tag{4.9}$$

For any (finite) cutoff scale Λ, any chosen vacuum configuration $\{\langle T\rangle, \langle A\rangle, \langle D\rangle\}$ can be reproduced by choosing the values of the parameters as:

$$m^2 = 2\lambda\langle T\rangle \ , \quad \lambda_A = -2\frac{\langle A\rangle}{\Lambda^4} < 0 \ , \quad \mu = \frac{\langle D\rangle}{\Lambda^2} \ . \tag{4.10}$$

For the validity of the effective theory approach, and to reproduce the hierarchical Yukawas, the vacuum expectation values should be smaller than the corresponding power of the scale Λ. This means, in particular, that μ and λ_A are required to be quite suppressed also in this scenario.[7] Such suppression is directly responsible for the Yukawa hierarchy, and this does not represent the kind of natural explanation we are looking for. Notice also that, as $\lambda_A < 0$ and $\mu > 0$, the minimum of the tree-level potential corresponds to the symmetric configuration $\xi_1 = \xi_2 = \xi_3$ so that ΔV does not represent a small perturbation of the tree level vacuum structure. The correction to the potential can have such a large effect only because in this scenario the parameters of the tree-level potential are assumed to have values much smaller than the size of the corrections.

In the previous discussions, the scale dependence of the Lagrangian parameters has been disregarded, although their RGE running can add new features to the minimization problem. For example, although the functional form of the SM Higgs potential, $V(h) = -(m^2/2)h^2 + (\lambda_h/4)h^4$, allows for only one minimum, its renormalization-improved version, with running m^2 and λ_h evaluated at a renormalization scale $Q \sim h$ (as necessary for a faithful description of the potential in a large range of h values) has a richer structure. In fact, for low values of the Higgs mass (as the value currently measured at the LHC, $m_h \simeq 126$ GeV [24]) a second minimum develops at a scale much larger than the electroweak scale, when the quartic Higgs coupling λ_h is driven to negative values due

[6] A potential of this form is given in appendix B of [11].

[7] For example, by adding to V_A, eq. (2.14), higher order corrections in the form $(\lambda_A + \langle T\rangle/\Lambda^2 + \langle T\rangle^2/\Lambda^4 + \ldots)A$, we would get from the hierarchy condition $\langle T\rangle^2 > \langle A\rangle$ and from the second relation in eq. (4.10) that the tree level coupling λ_A must be smaller than the second order correction $\sim \langle T\rangle^2/\Lambda^4$.

to large radiative corrections from the top Yukawa coupling (see ref. [25] for the state-of-the-art analysis). For particular values of the Higgs mass, the negative quartic stays small enough to cancel against the value of its β function, providing a second solution to the minimization equation $dV/dh \simeq [\lambda_h + (1/4)d\lambda_h/d\log h]h^3 = 0$. We could imagine something similar happening with the potential for the Yukawa fields Y: the structure of this potential at the renormalization scale relevant for the largest nonzero vev $Q^2 \sim \xi_3$ could in principle be different from its structure at the lower scales relevant for the smaller vevs ξ_1 and ξ_2. However, the problem is now complicated by the fact that the potential is a multifield one and the correct description of a hierarchical vacuum requires the use of three different renormalization scales simultaneously (a multiscale problem often faced in effective potential studies, see *e.g.* [23]). We limit ourselves to pointing out this possibility as worthy of future exploration.

5 Symmetry breaking via reducible representations

The results of the previous sections make clear which way is left open to get a phenomenologically viable pattern of vevs for the components of the Yukawa field Y. Namely, the flavour symmetry must be broken down to H_{abc}, eq. (2.24), already at the tree level. For this, we need a non-minimal set of scalar fields in reducible representations of the flavour group. In fact, breaking a symmetry by means of reducible representations avoids at once the issue of maximal stability little groups: even defining what are *open strata* and *closed strata* is not clear in this case. A minimal enlargement of the scalar sector involves adding two multiplets, $Z_{L,R}$, that transform respectively in the fundamental of one of the two group factors $SU(3)_L \times SU(3)_R$ while they are singlets under the other one:[8]

$$Z_L = (\mathbf{3}, \mathbf{1}), \qquad Z_R = (\mathbf{1}, \mathbf{3})\,. \tag{5.1}$$

The most general $SU(3)_L \times SU(3)_R$ invariant potential involving Z_L, Z_R and $Y = (\mathbf{3}, \bar{\mathbf{3}})$ can be written as

$$V = V_A + V_{\mathcal{D}} + V_l + V_m + V_{\tilde{\nu}}\,, \tag{5.2}$$

with

$$\begin{aligned} V_l &= \lambda\left(T - \frac{m^2}{2\lambda}\right)^2 + \lambda_L\left(|Z_L|^2 - \frac{m_L^2}{2\lambda_L}\right)^2 + \lambda_R\left(|Z_R|^2 - \frac{m_R^2}{2\lambda_R}\right)^2 \\ &\quad + g\left[\left(T - \frac{m^2}{2\lambda}\right) + \frac{g_{1L}}{g}\left(|Z_L|^2 - \frac{m_L^2}{2\lambda_L}\right) + \frac{g_{1R}}{g}\left(|Z_R|^2 - \frac{m_R^2}{2\lambda_R}\right)\right]^2, \qquad (5.3) \\ V_m &= g_{2L} Z_L^\dagger Y Y^\dagger Z_L + g_{2R} Z_R^\dagger Y^\dagger Y Z_R\,, \qquad (5.4) \\ V_{\tilde{\nu}} &= \tilde{\nu}\, Z_L^\dagger Y Z_R + \text{h.c.}\,, \qquad (5.5) \end{aligned}$$

[8] The vevs of $Z_{L,R}$ represent sources of flavour breaking that do not transform as the SM Yukawa couplings, and thus in principle imply a non Minimal Flavour Violating (MFV) [26] scenario. However, by a suitable choice of the representations of the messengers that generate the Yukawa operators, one can forbid all dangerous FCNC Yukawa-like operators involving $Z_{L,R}$.

while V_A and $V_{\mathcal{D}}$ have already been given in eq. (2.14) and eq. (2.15). V_l in eq. (5.3) can be equivalently written as

$$V_l = \hat{\lambda}\left(T - \frac{\hat{m}^2}{2\hat{\lambda}}\right)^2 + \hat{\lambda}_L\left(|Z_L|^2 - \frac{\hat{m}_L^2}{2\hat{\lambda}_L}\right)^2 + \hat{\lambda}_R\left(|Z_R|^2 - \frac{\hat{m}_R^2}{2\hat{\lambda}_R}\right)^2 + 2T\left(g_{1L}|Z_L|^2 + g_{1R}|Z_R|^2\right) + 2\,\frac{g_{1L}\,g_{1R}}{g}\,|Z_L|^2\,|Z_R|^2 + \text{const}\,. \tag{5.6}$$

This second way of writing V_l makes apparent which relevant terms quadratic and quartic in the fields have been included; however, eq. (5.3) is more convenient for minimization, since it makes transparent that V_l (which for $\lambda, \lambda_L, \lambda_R, g > 0$ has manifestly its minimum when it vanishes) determines the 'lengths' of the three multiplets to be, respectively:

$$\langle T\rangle \equiv v_Y^2 = \frac{m^2}{2\lambda}, \qquad \langle|Z_L|^2\rangle \equiv v_L^2 = \frac{m_L^2}{2\lambda_L}, \qquad \langle|Z_R|^2\rangle \equiv v_R^2 = \frac{m_R^2}{2\lambda_R}\,, \tag{5.7}$$

without having other effects on the particular structure and/or alignment of the three vevs. Note in particular that eq. (5.3) also makes apparent that the correct tree-level minima can already be obtained from the first line alone, that is, by setting g_{1L}, g_{1R} and $g \to 0$ (in particular, this limit largely simplifies the identification of the Goldstone bosons).

The role of V_m is instead that of misaligning (or aligning) the vevs of $Z_{L,R}$ with the vev of Y. Our aim is to enforce a maximum misalignment, in order to obtain the smallest possible intersection among the little groups of the vevs of the three multiplets, and this is achieved for $g_{2L}, g_{2R} > 0$. Thus we assume that both couplings in V_m are positive, and we will not analyze other possibilities, since they would only yield phenomenologically uninteresting vacuum structures.

Once the matrix of constant fields Y has been brought to the diagonal and partially real form, eq. (2.16), by means of symmetry rotations, we are left with the freedom of removing only one phase from Z_L and another one from Z_R. However, we will now argue that, for the minimization problem, we can take $Z_{L,R}$ real. After defining $\nu = |\tilde{\nu}|$, the last term $V_{\tilde{\nu}}$ can be rewritten as:

$$V_{\tilde{\nu}} = 2\,\nu\,\left|Z_L^\dagger Y Z_R\right|\cos\phi_{LR}\,. \tag{5.8}$$

At fixed values of $\langle Y\rangle$, $\langle Z_L\rangle$, $\langle Z_R\rangle$ the minimum of $V_{\tilde{\nu}}$ occurs when $\cos\phi_{LR} = -1$, that is, for real and negative values of the vev of the trilinear term $\langle Z_L^\dagger Y Z_R\rangle$. On the other hand, it is easy to verify that the modulus of the trilinear term in eq. (5.8) is extremized for real values of the three vevs $\langle Y\rangle$, $\langle Z_L\rangle$, $\langle Z_R\rangle$. We have already argued that minimization of the potential can be carried out by taking $V_{\mathcal{D}}$ in the simplified form V_D given in eq. (2.17), which corresponds to real Y. From eq. (5.8) we can similarly conclude that, after including $Z_{L,R}$, the minimization of the potential can be explored around the configuration

$$V_{\tilde{\nu}}^{\text{min}} \equiv V_\nu = -2\,\nu\,Z_L^\dagger Y Z_R\,, \tag{5.9}$$

where $Z_{L,R}$ can also be taken to be real. With this simplification it is not difficult to work out the structure of the vevs at the minimum of the potential. Let us start by setting

$\nu \to 0$. For g_{2L}, $g_{2R} > 0$ and Y in diagonal form, V_m is always positive, and thus it is minimized when it vanishes, which occurs when the vevs of $Z_{L,R}$ are misaligned with respect to the vev of Y, as for example $\langle Y\rangle = v_h\,\mathrm{diag}\,(0,0,1)$, $\langle Z_L\rangle = v_L\ (c_L, s_L, 0)$ and $\langle Z_R\rangle = v_R\ (c_R, s_R, 0)$, with $c^2_{L,R}+s^2_{L,R} = 1$. Thus, in this limit the minimum of the potential is $V^{\rm min}_{(\nu=0)} = 0$. However, when $\nu \neq 0$ lower (negative) values of V_ν become possible: for small but nonvanishing values of the first two diagonal entries in $\langle Y\rangle$ the negative sign of V_ν implies the possibility of adding a negative contribution to the minimum of the potential. It is true that there is also a price to pay since the minimum of V_m will then be lifted to positive values, but while this effect in V_m is quadratic in the small $\langle Y\rangle$ entries, it is linear, and thus dominant, in V_ν. We thus expect that V_ν can favour configurations in which the zero entries in $\langle Y\rangle$ are lifted to non-zero values.

To see explicitly how this can occur, let us take the new vacua in the form:

$$\begin{aligned}
&\langle Y\rangle = v_Y\,\mathrm{diag}\,(\epsilon,\epsilon',y)\,, \quad &&\text{with} \quad \epsilon,\epsilon' \ll y\,, \quad &&\epsilon^2+\epsilon'^2+y^2=1,\\
&\langle Z_L\rangle = v_L\ (z_L,\epsilon'_L,\epsilon_L) \quad &&\text{with} \quad \epsilon'_L,\epsilon_L \ll z_L, \quad &&{\epsilon'_L}^2+\epsilon_L^2+z_L^2=1,\\
&\langle Z_R\rangle = v_R\ (z_R,\epsilon'_R,\epsilon_R) \quad &&\text{with} \quad \epsilon'_R,\epsilon_R \ll z_R, \quad &&{\epsilon'_R}^2+\epsilon_R^2+z_R^2=1\,.
\end{aligned} \tag{5.10}$$

V_l in eq. (5.3) fixes the 'lengths' v_Y and $v_{L,R}$, and vanishes. So we need to consider only the effect of

$$\begin{aligned}
V_\epsilon &\equiv V_A + V_D + V_m + V_\nu\\
&= \lambda_A A - 2\,\mu\,D + g_{2L} Z_L^\dagger Y Y^\dagger Z_L + g_{2R} Z_R^\dagger Y^\dagger Y Z_R - 2\,\nu\, Z_L^\dagger Y Z_R\,,
\end{aligned} \tag{5.11}$$

which vanishes in the limit $\nu \to 0$ but remains of $\mathcal{O}(\epsilon)$ when $\nu \neq 0$. Plugging into V_ϵ the vevs in eq. (5.10) we obtain:

$$\begin{aligned}
\frac{1}{v_Y^2} V_\epsilon = {}& \lambda_A v_Y^2 \left[\epsilon^2\epsilon'^2 + y^2\left(\epsilon'^2+\epsilon^2\right)\right] - 2\,v_Y\,\mu\,\epsilon\epsilon' y\\
&+ g_{2L}\,v_L^2\,\left(z_L^2\epsilon^2 + \epsilon_L'^2\epsilon'^2 + \epsilon_L^2\,y^2\right) + g_{2R}\,v_R^2\,\left(z_R^2\epsilon^2 + \epsilon_R'^2\epsilon'^2 + \epsilon_R^2 y^2\right)\\
&- 2\,\nu\,\frac{v_L v_R}{v_Y}\,\left(z_L z_R \epsilon + \epsilon'_L\epsilon'_R\epsilon' + \epsilon_L\epsilon_R\,y\right)\,.
\end{aligned} \tag{5.12}$$

The role of V_ν, which is linear in ϵ in preferring values of $\epsilon \neq 0$ is apparent, as well as the role of V_D in favouring in turn $\epsilon' \neq 0$. A simple illustrative solution in terms of the relevant parameters μ and ν can be obtained by setting for simplicity $v_Y = v_L = v_R$ and $\lambda_A = g_{2L} = g_{2R}$ and by recalling that the lengths are fixed (i.e. $y^2 = 1-\epsilon^2-\epsilon'^2$ etc.). Solving for the extremal conditions $\partial V_\epsilon/\partial\epsilon = \partial V_\epsilon/\partial\epsilon' = \partial V_\epsilon/\partial\epsilon_{L,R} = \partial V_\epsilon/\partial\epsilon'_{L,R} = 0$, and truncating to terms $\mathcal{O}\left(\epsilon^2\right)$ we obtain a unique solution for the global minimum:

$$\epsilon = \frac{\lambda_A\,\nu\,v_Y}{3\lambda_A^2\,v_Y^2 - \mu^2}\,, \qquad \epsilon' = \frac{\mu}{\lambda_A\,v_Y}\,\epsilon\,, \qquad \epsilon_{L,R} = \epsilon'_{L,R} = 0, \tag{5.13}$$

with $V_\epsilon^{\rm min} = -\nu\,v_Y^3\,\epsilon$. The relations in eq. (5.13) show that phenomenologically acceptable vevs for the up-quark Yukawa couplings could be obtained for example for $\nu \sim \mu \sim 10^{-2}\cdot\lambda_A\,v_Y$, yielding $\epsilon' \sim 10^{-2}\cdot\epsilon \sim 10^{-4}$. A second possibility, namely $\nu \sim \lambda_A\,v_Y \sim 10^{-2}\cdot\mu$ yielding $\epsilon \sim 10^{-2}\cdot\epsilon' \sim 10^{-4}$ is in fact not viable since in this case it is not possible to satisfy

simultaneously the constraint on μ/m from eq. (2.21) and the condition $v_Y \simeq m/\sqrt{2\lambda} \sim \Lambda$ from the value of the top-quark Yukawa coupling.

To verify the correctness of the simple minimization procedure that we have outlined above, we have also performed a set of numerical minimizations. This has confirmed the lifting of the zeroes in $\langle Y \rangle$ to nonvanishing entries whose values, among other things, are controlled in a crucial way by ν and μ that, in order to reproduce the observed Yukawa hierarchies, should be somewhat suppressed with respect to the other dimensional parameters m, m_L, m_R. Let us stress however, that the numerical analysis that we have carried out had just the scope of confirming the structures obtained analytically, and that the extremum corresponds to a real global minimum, and did not aim at a thorough exploration of the full parameter space for finding sets of values satisfying particular naturalness requirements.

One final interesting point is that in all cases we find that two entries in both $\langle Z_{L,R} \rangle$ vanish, see eq. (5.13). The little group of the vevs of the two fundamental representations is then $H_{LR} = SU(2)_L \times SU(2)_R$. The intersection of H_{LR} with the little group of the vev of the bi-fundamental $H_Y = U(1)_{(\lambda_3)_{L+R}} \times U(1)_{(\lambda_8)_{L+R}}$ is then $H_{LR} \cap H_Y = U(1)_{(\sigma_3)_{L+R}}$, corresponding to the diagonal generator $\sigma_3 = \frac{1}{2}(\sqrt{3}\lambda_8 - \lambda_3)$ of $SU(2)_{L+R}$. This means that, out of the $8 + 8$ generators of the flavour group, only one remains unbroken, and we can then predict fifteen Goldstone bosons. We have studied numerically the spectrum of scalar particles, and this confirmed that the symmetry breaking pattern induced by scalars in the reducible representation $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \bar{\mathbf{3}})$ is indeed $SU(3)_L \times SU(3)_R \to U(1)_{(\sigma_3)_{L+R}}$.

6 Conclusions

The possibility that the highly non-symmetric spectra of the fermions with the same SM quantum numbers could arise from the specific structure of the vacuum of an otherwise flavour-symmetric theory, is theoretically very attractive. In this paper, we have studied which premises are needed to render this idea phenomenologically viable.

In ref. [12] it was found, as a promising starting point, that an $SU(3)_L \times SU(3)_R$ invariant tree-level potential for Yukawa fields Y transforming as the irreducible bi-fundamental $(\mathbf{3}, \bar{\mathbf{3}})$ representation of the flavour group admits vacuum structures $\langle Y \rangle \propto \mathrm{diag}(0, 0, 1)$. It was then natural to ask if the vanishing entries could be lifted to suppressed but nonvanishing values by some type of small perturbation. This would correspond to a stepwise breaking of the flavour symmetry.

In this paper we have argued that the structure of the tree-level vacua is perturbatively stable. Regarding the possibility of a stepwise symmetry breaking due to loop corrections, it was already discussed long ago [19, 20] that this can only occur in the presence of additional (non NGB) scalars that are massless at tree level. By direct computation of the effective potential (see the Appendix) we have confirmed that the little groups of the vacua of the tree-level potential are also the little groups of the vacua of the one-loop corrected effective potential, and that this remains true even in the presence of additional scalars that are *unnaturally* massless (that is, massless due to fine tuning of some parameters). We have also argued that this result can be straightforwardly extended to the possible

effects of operators of higher dimension. In Section 4 we have further confirmed this and argued that, except for some special cases, a generic $SU(3)_L \times SU(3)_R$ invariant function of a single scalar field Y transforming in the bi-fundamental representation of the group, admits vacuum structures $\langle Y \rangle$ with at least two equal eigenvalues, and thus it cannot yield a fully hierarchical pattern.

We have thus learned that a phenomenologically viable Yukawa structure $\langle Y \rangle \propto \mathrm{diag}(\epsilon, \epsilon', 1)$ must arise already at the tree level, and that in order to achieve this, the breaking must occur via reducible representations. After enlarging the scalar sector by adding two multiplets Z_L and Z_R transforming respectively in the fundamental representation of the $SU(3)_{L,R}$ factors of the flavour group, we have constructed the most general fourth-order potential involving $Z_{L,R}$ and Y, and we have shown that minima yielding the hierarchical structure $\langle Y \rangle \propto \mathrm{diag}(\epsilon, \epsilon', 1)$ can indeed appear. The Yukawa hierarchy for the up-type quarks can then be qualitatively reproduced at the cost of a relatively mild hierarchy between the dimensional parameters of the scalar potential, not exceeding $\sim 10^{-2}$. The hierarchies in the down-quark and charged lepton sectors are also reproducible with even milder hierarchies in the fundamental parameters.

In a more complete scenario we would first need to extend the symmetry to the full quark flavour group $SU(3)_{Q_L} \times SU(3)_{u_R} \times SU(3)_{d_R}$, and then couple through appropriate renormalizable invariant terms the Yukawa fields of the up and down-quark sectors Y^q, Z^q ($q = u, d$) and Z^Q (see ref. [12] for a first attempt with only irreducible representations Y^q). Besides reproducing the mass hierarchies, such a scenario should also reproduce the hierarchies in the CKM mixing angles, and yield a nonvanishing value for the CP violating Jarlskog invariant [27]. We believe that, in spite of its complexity, such a program can be carried out successfully, and we expect to publish soon some results that go in this direction [28].

Eventually, one should also worry about the testability of this type of construction. It is clear that the scalar potential of a more complete model will contain a number of fundamental parameters much larger than the number of observables (the six quark masses, the three mixing angles, and the CP violating phase δ) implying that it is unlikely that predictive relations among different observables could arise. Direct evidences might arise from the fact that if the flavour symmetry is global, then SSB implies the presence of Nambu-Goldstone bosons that could show up in yet unseen hadron decays or in rare flavour violating processes [29]. If the flavour symmetry is instead gauged [9, 10], then to ensure the absence of gauge anomalies additional fermions must be introduced [10], and their detection could represent a smoking gun for this type of models. All this remains, however, a bit speculative, especially because the theory provides no hint of the scale at which the flavour symmetry gets broken, and very large scales would suppress most, if not all, types of signatures. In spite of these considerations, being able to reproduce the observed pattern of Yukawa couplings from the SSB of the flavour symmetry would certainly represent an important theoretical achievement, and we believe that the results discussed here can provide some relevant steps in this direction.

A Effective potential for a single irreducible representation

The $SU(3)_L \times SU(3)_R$ invariant one-loop Coleman-Weinberg (CW) [21, 30] effective potential for a scalar field Y in the bi-fundamental representation $(\mathbf{3}, \overline{\mathbf{3}})$ can be written, in the $\overline{\text{MS}}$ scheme, as:

$$V = V_0 + V_1 \tag{A.1}$$

with

$$V_0 = \lambda \left[T(Y) - \frac{m^2}{2\lambda} \right]^2 + \lambda_A A(Y) + \mu \mathcal{D}(Y) + \mu^* \mathcal{D}^*(Y) \tag{A.2}$$

$$V_1 = \frac{1}{64\pi^2} \sum_i M_i^4(Y) \left[\log \frac{M_i^2(Y)}{\Lambda^2} - \frac{3}{2} \right] \tag{A.3}$$

where all the parameters in V_0 are renormalized at the scale Λ: $\lambda = \lambda(\Lambda)$, $m^2 = m^2(\Lambda)$, etc. The field dependent mass functions $M_i^2(Y)$ in eq. (A.3) are the eigenvalues of the matrix

$$[\mathcal{M}^2]_{ij,kl} = \left. \frac{\partial^2 V_0}{\partial \mathcal{Y}_{ij} \partial \mathcal{Y}_{kl}} \right|_Y \tag{A.4}$$

where $\mathcal{Y}_{ij} = \{R_{ij}, J_{ij}\}$ with $R_{ij}\,(J_{ij}) = \sqrt{2}\,\text{Re(Im)} Y_{ij}$ and, without loss of generality, we can take the background constant field Y in the diagonal form eq. (2.16). Here we are including only the contributions to the effective potential that come from scalar self-interactions between the components of the Y field. In concrete models Y will interact with other sectors of the theory, which will then also contribute to the loop corrected potential.

From eq. (A.4) one can compute straightforwardly the two traces

$$\text{Tr}\mathcal{M}^2 = 8\,(5\lambda + \lambda_A)\,T - 18m^2, \tag{A.5}$$

$$\begin{aligned} \text{Tr}\mathcal{M}^4 = {} & 4T^2 \left[(5\lambda + \lambda_A)^2 + (\lambda - \lambda_A)^2 \right] - 8T \left[2m^2\,(5\lambda + \lambda_A) - |\mu|^2 \right] \\ & + 48\lambda\lambda_A A + 24\,(\lambda + \lambda_A)\,(\mu\mathcal{D} + \mu^*\mathcal{D}^*) + 18m^4. \end{aligned} \tag{A.6}$$

which give the field dependent divergent part of the CW potential, which (using a cutoff regularization) reads:

$$\delta_\Lambda V = \frac{\Lambda^2}{32\pi^2}\,\text{Tr}\mathcal{M}^2 + \frac{\log\Lambda^2}{64\pi^2}\,\text{Tr}\mathcal{M}^4\,. \tag{A.7}$$

The logarithmic part can be used to obtain the beta functions of λ, λ_A, m^2 and μ (up to contributions from other sectors of the theory).

Computing the finite contribution to the CW potential eq. (A.3) is instead a difficult problem, since it requires diagonalizing the full 18×18 matrix eq. (A.4), namely solving the eigenvalue equation

$$\det(M^2 \cdot I - \mathcal{M}^2) = 0\,, \tag{A.8}$$

where M^2 are the eigenvalues, $I \equiv I_{18\times 18}$ and $\mathcal{M}^2$ is given in eq. (A.4) evaluated at $Y = \langle Y \rangle = \text{diag}(\sqrt{\xi_1}, \sqrt{\xi_2}, \sqrt{\xi_3})$.

The problem is somewhat simplified by the fact that the eigenvalues will come in multiplets of the unbroken little group, and another simplification is obtained by looking for a minimum around the configuration eq. (2.18), that is $\langle Y \rangle = \mathrm{diag}(\sqrt{\xi_1}, \sqrt{\xi_2}, \sqrt{\xi_3})$.[9] We have managed to solve the problem by means of a 'brute force' procedure, that can be resumed in the following steps:

(i) The determinant in eq. (A.8) admits a factorization of the form

$$\det(M^2 \cdot I - \mathcal{M}^2) = P^{(6)}(M^2) \times \Pi_{i=1}^{3} \Pi_{\pm} (M^2 - M_{i\pm}^2)^2 \,, \tag{A.9}$$

where $P^{(6)}(M^2)$ is a sixth-order polynomial in M^2. This factorization allows to identify the first twelve eigenvalues (labeled with $i\pm$ with $i = 1, 2, 3$) that come arranged into six degenerate doublets, and are:

$$M_{i\pm}^2 = \overline{m}^2 + \frac{1}{2} \lambda_A \, \xi_i \pm \sqrt{F\left(\xi_i, T, A, D\right)}, \tag{A.10}$$

where

$$\overline{m}^2 \equiv -m^2 + 2\lambda T, \tag{A.11}$$

$$F\left(\xi_i, T, A, D\right) = \frac{1}{2} \left[{\lambda_A}^2 \left(\frac{\xi_i}{2} - T \right) + \mu^2 \right] \xi_i + \lambda_A \left(\lambda_A A + 2\mu D \right). \tag{A.12}$$

The eigenvalues ξ_i of the matrix of constant classical fields $YY^\dagger$, and can be explicitly written in terms of the invariants T, A, D by solving the cubic equation:

$$\det(\xi I_{3\times 3} - YY^\dagger) = \xi^3 - T\xi^2 + A\xi - D^2 = 0. \tag{A.13}$$

(ii) Concerning the roots of $P^{(6)}(M^2)$ in eq. (A.9), it is well known that there is no formula in radicals to solve polynomial equations beyond quartic [31]. However, on physics grounds we know that $P^{(6)}(M^2) = 0$ must be solvable with real and positive solutions. We have then approached the problem of extracting the solutions by studying various limits with increasing steps of complexity: $\lambda_A, \mu = 0$ ($\lambda \neq 0$); $\lambda, \mu = 0$ ($\lambda_A \neq 0$); $\mu = 0$ and λ, $\lambda_A \neq 0$; $\lambda_A = 0$ and λ, $\mu \neq 0$. This allowed us to identify some characteristic structures appearing in the solutions.

(iii) Finally, given that $P^{(6)}(M^2) = 0$ is solvable, it follows that the sixth order polynomial must be factorizable in several different ways into lower order polynomials like $P^{(3)} \cdot P^{(3)}$ or $P^{(2)} \cdot P^{(2)} \cdot P^{(2)}$ or $P^{(2)} \cdot P^{(4)}$, which all must have real and positive solutions. This implies that the structure of the solutions can be recast in the general form of

[9] This is justified by the fact that after reabsorbing the phase of $\tilde{\mu}$ in $\mathcal{D}$, any imaginary part of the determinant, corresponding in our basis to $J_{33} \neq 0$, would imply deviation from $\cos\phi_{\mathcal{D}} = -1$ and would shift the value of $V_{\mathcal{D}}$ eq. (2.15) away from its minimum. In the loop corrections, functions involving terms like $\mathcal{D}^n + \mathcal{D}^{*n} \sim D^n \cos(n\,\phi_{\mathcal{D}})$ also appear. For even n they are extremized at $\phi_{\mathcal{D}} = \pi/n$ and could thus shift, at least in principle, the minimum from π to the doubly degenerated point $\phi_{\mathcal{D}} = \pi \pm \alpha$ (with suitable values of α), yielding a complex $\langle \mathcal{D} \rangle$ and spontaneous breaking of CP. However, the loop coefficient suppressing these contributions is small enough to guarantee stability of the minimum in π and that the background value of the determinant does not acquire an imaginary part. This is of course in agreement with ref. [20] where the necessary conditions for stepwise breaking of discrete symmetries were stated.

solutions of quadratic, cubic and quartic polynomial equations. When written down as the roots of two cubic polynomials the remaining six eigenvalues read (with $i, i' = \{1, 2, 3\}$):

$$\begin{aligned} M_i^2 &= \overline{m}^2 + \frac{2}{3}\lambda_A T + \frac{r_i}{3} P^{1/3} + \frac{1}{3 r_i} \frac{Z_P}{P^{1/3}} \ , \\ M_{i'}^2 &= \overline{m}^2 + \frac{2}{3}\lambda_A T + \frac{4}{3}\lambda T + \frac{r_{i'}}{3} Q^{1/3} + \frac{1}{3 r_{i'}} \frac{Z_Q}{Q^{1/3}} \ , \end{aligned} \tag{A.14}$$

where $r_{i,i'}$ are the three roots of $x^3 + 1 = 0$, that is, $\{-1, e^{-i\pi/3}, e^{i\pi/3}\}$, while

$$\begin{aligned} Z_P &= \lambda_A^2 \left(T^2 - 3A\right) + 3\mu^2 T, \\ Z_Q &= Z_P + 16\lambda^2 T^2 + 4\lambda_A \left[2\lambda \left(9A - T^2\right) + 3\lambda_A A\right] + 36 \left(2\lambda + \lambda_A\right) \mu D, \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} P = & \frac{1}{2}\sqrt{X_P^2 - 4 Z_P^3} - 9\lambda_A \mu^2 T^2 - \frac{9}{2}\lambda_A^3 A T + \lambda_A^3 T^3 \\ & + \frac{27}{2}\left[2\mu^3 D + 2\lambda_A \mu^2 A + \lambda_A^3 D^2\right], \\ Q = & -\frac{1}{2}\sqrt{X_Q^2 - 4 Z_Q^3} + 9\left(4\lambda - \lambda_A\right)\mu^2 T^2 - \left(4\lambda - \lambda_A\right)^3 T^3 \\ & - \frac{27}{2}\left[32\lambda^2 \mu D + 32\lambda\lambda_A \left(\lambda A + \mu D\right) - \lambda_A \left(4\lambda A - 8\mu D + \lambda_A A\right)\right] T \\ & - \frac{27}{2}\left[2\mu^3 D + 16\lambda\mu^2 A + 2\lambda_A \mu^2 A + 27\lambda_A^2 \left(4\lambda + \lambda_A\right) D^2\right], \end{aligned} \tag{A.16}$$

with

$$\begin{aligned} X_P = & \ 54\mu^3 D - 18\lambda_A \mu^2 \left(T^2 - 3A\right) + \lambda_A^3 \left(2T^3 - 9AT + 27D^2\right), \\ X_Q = & \ 2\left[64\lambda^3 T^3 - 36\lambda\mu^2 T^2 + 432\lambda^2 \mu D T + 27\mu^2 \left(8\lambda A + \mu D\right)\right] \\ & + 6\lambda_A \left[-16\lambda^2 T^3 + 3\mu^2 T^2 + 144\lambda\mu D T + 9A\left(16\lambda^2 T + \mu^2\right)\right] \\ & + 3\lambda_A^2 \left(-36\lambda A T + 8\lambda T^3 + 972\lambda D^2 + 72\mu D T\right) \\ & - \lambda_A^3 \left(27AT + 2T^3 - 729D^2\right) \ . \end{aligned} \tag{A.17}$$

By solving the eigenvalue problem numerically, and confronting the solutions with the corresponding numerical values of the analytic expressions for M_i^2, we have verified the correctness of the analytic formulas given in eqs. (A.10)-(A.14).

This explicit expression for the potential can then be used to explore the change of the tree-level vacuum induced by the radiative corrections. An alternative approach (see *e.g.* [32]) that does not require solving for the exact mass eigenvalues that enter the CW potential is to calculate the field derivatives of the potential (or derivatives with respect to the invariants T, A, D) needed in the minimization equations using perturbation theory around the tree-level vacuum, obtaining the derivatives of the masses directly from the derivatives of the characteristic polynomial $\det(M^2 \cdot I - \mathcal{M}^2) = 0$ (which can be written in terms of the T, A, D invariants, although we do not write those expressions explicitly). We have also verified in this manner that the one-loop corrections shift the tree-level minimum rotating it in field space, but without changing its minimum structure, which still yields a Yukawa coupling matrix of the form $\langle Y \rangle \propto \mathrm{diag}(0, 0, 1)$.

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